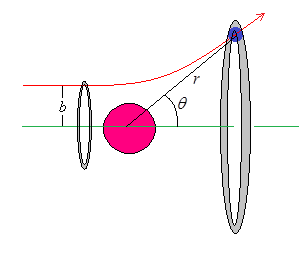
**Relating Scattering Formalism to TDPT**

So in the beginning we developed a time-independent quantum mechanical approach to scattering, then developed a perturbative approach formulated in terms of a T-matrix, which could be put in terms of an S-matrix that was ultimately relatable to the time-development S-matrix. Now we’d like to go backwards sort of and start with time-dependent quantum mechanics, and reconstruct the scattering formulas. This is the more natural/intuitive approach, though slightly more technical. Anyway, let’s look at the connection between this and the differential scattering cross-section.



**Method 1**

Consider a Gaussian-ish wavepacket, |ψ0(p)>, with momentum p, and centered on the z-axis. And we can move the wavepacket in the transverse direction to give it an impact parameter b, via |ψb(**p**)> = exp(i·**b**)|ψ0(**p**)>. Now let’s say we send in a beam of such particles, evenly distributed throughout an area, Area, and consequent scattering cross-section σ. As discussed in the scattering file, we can write the differential scattering cross-section as:



Filling in what the probability is, we have:



where S is the time-development S-matrix, whose matrix elements are given by, in the long-time limit (see adiabatic perturbation in TDPT),



Recall the magnitude and direction of **p** is fixed by us. But only the direction of **p´** is fixed, to be along the direction (Ω, Ω+dΩ). Continuing,



Now we will consider only non-forward scattering Ω’s, as emphasized elsewhere. So we can replace S by T etc.



If we extend the beam Area to ∞, then we may do the b integral. If we had a finite size beam, this would still be warranted as long as Area >> σT. Likely, the error involved in taking the limit b → ∞, rather than leaving it finite, goes as exp(-b/R), where R is the range of the potential. So if b >> R, then we have no problem taking that limit. But for inifinite ranged potentials, I suspect the error involved is more like exp(-bsinθ/R), where R is ‘some’ length scale. And then we can see that even if b is large, we can take the infinite limit for most angles, but cannot take the infinite limit for small θ’s approaching zero without invoking significant error. So I think we’d have to merely acknowledge that we don’t know what happens for small angles. There surely isn’t an abrupt cut-off because all angles are possible outcomes, but it probably does drop to a finite # or zero. The θmin, approximately where we cannot trust our formula, would therefore be b-dependent in some fashion, like θmin ~ 1/b. Perhaps it could be roughly estimated using classical mechanics. We should expect, in the b → ∞ limit, ∫(dσ/dΩ)dΩ to give us σT for finite ranged potentials, and I suppose the beam Area elsewise. Anyway, so then we’d get:



Now I think we take the limit that |ψ0(**p**)> is rather highly peaked about the momentum **p**, so that it basically acts as a delta function, and we can write:



And because p´´ and p´´z are the same since **p** lies along the z-axis, we come to:



which we have of course seen before.

**Method 2**

Perhaps a quick(er) and dirty derivation would be to use box normalized states: |kbox> = (1/√V)ei**k**·**x**. Then consider the transition probability between two box states k and k´. We’d like to parler that into a formula for the transition probability between k and a differential solid angle about k´. So consider (kind of setting ℏ → 1):



where ρ(E´) is the density of states per unit volume, and the dΩ integral is over the entire solid angle, but divided by 4π so as not to overcount states (b/c ρ(E) already counts them all). From Condensed Matter Theory file (free Electrons), we have (not including spin degeneracy):



So then:



Now we need to get P(**k**→**k**´). From TDPT (scattering perturbation), we have for the *rate* of scattering from |kbox> to |k´box> (because the TDPT formula was general and didn’t presume any specific type of state):



We have to convert this to a *probability* of scattering, and that can be done by dividing by the rate of particles incomming. This would be Γ(k) = ρAv, where ρ = particle density, which is 1/V for our wavefunction. A is cross section area, which is A = V2/3, and v = k/m is velocity of the particle. And so can say,



Filling this into P(k→dΩ) we have:



Now we also have:



where A is the scattering cross section of our beam (because that *is* the area of our beam). So equating the two, we have:



Now filling in the density of states:



And now filling in our expression for Γ(k→k´), we have:



But now let’s put T in terms of delta normalized states instead of box normalized states. So we’ll note that|kbox> = √((2π)3/V)|kdelta >. Now letting k represent delta normalized states, we have:



which is what we got above, sans some ℏ factors I’ve been neglecting.

**Formula for total cross-section / optical theorem**

From the last method, we can see from one of the first few lines, that we’d expect the total cross section is something like the scattering rate integrated over all destination states. One of the ways we can make this connection explicit is as follows (and we’ll also provide a connection between the off-shell T-matrix encountered in the constant TDPT file and the on-shell T-matrix first encountered in the scattering TDPT file, as well as in this scattering folder obviously). So start with the off-shell T-matrix:



where



The last line in the T-equation can be motivated as follows,



of course we played loose with the fact that these G’s and H’s are operators, not numbers per se´, but if one is more careful we can show the same thing. So now consider boxing our T matrix against two delta normalized k states <k|T(ω)|k>. We get:



Now insert resolution of identity in terms of the eigenstates of H, which we’ll call |>,



where k are the eigenenergies of H. And now revisiting the argument we made in the RSPT 3D scattering file….



where here is the onshell scattering matrix and in the last line we used the general result,



So now we can say Tkk(ω) is equal to:



So can now explicitly see that the poles of Tkk(ω) are the eigenenergies of the total Hamiltonian, as was argued in the constant perturbation TDPT file. Now I think that if the potential isn’t attractive enough to form bound states, then all the new energies should just be the old ones, so that while the wavefunctions may change, developing phase shifts and all, the energies remain the same (we saw this for instance when we looked at some of the repulsive potential scattering examples. So then we could say,



This is our connection between the off-shell scattering matrix on the left, and the on-shell matrix on the right. Using the identity (see Math Appendix),



we can write this as:



Let’s consider the imaginary part:



Say we evaluate this on-shell, that is at ω = εk. Then we have:



Of course Tkk(ω=εk) is just Tkk.



where we recognize Γ, from TDPT (scattering perturbation), as the total rate of scattering from out of our initial state |k>. And from the optical theorem in the RSPT 3D scattering file, we have:



So we can relate σ to the total scattering rate, Γ. But have to clean up some stuff first. So the Tk´k in the Γ/2 equation is implicitly evaluated between box states |kbox> (because we were kind of presuming discrete states in that TDPT formula), while the Tkk in σ is evaluated between delta normalized states, |k>. Since |k>box = eikr/√V = [(2π)3/2/√V]eikr/(2π)3/2 = [(2π)3/2/√V]|k>, we have:



and so we can equate,



and finally,



This makes sense on classical grounds. This is (1/V)vσ = ρvσ where ρ is the particle’s density basically, and v its velocity. And so this is the rate at which the particle is running into the scatterer.